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Abstract. In this paper we present a study of the restrictions imposed upon the motion of particles (material particles and photons) in the gravitational field of a rotating cylinder.

1. Introduction

The emission of photons from spherical bodies has been studied by Synge (1966). Motions of particles in a Kerr field have been investigated in detail by De Felice and Calvani (1972). Banerjee (1968) and Krori and Chaudhury (1978) have studied the escape of photons from cylindrical bodies. The study in the cylindrical case will not be complete without an investigation on the restrictions caused to the motions of particles (material particles and photons) by the rotation of cylindrical bodies about their axes. We have taken up this investigation in this paper.

2. Detailed calculations

First, we show that the velocity of a particle in the field of a rotating cylinder receives a contribution from the angular velocity $\mathrm{d}\phi/\mathrm{d}t$, where ϕ is a coordinate angle (see equation (6)).

Landau and Lifshitz (1971) have shown that the total energy of a particle of proper mass m_0 in a static gravitational field is given by

$$E = m_0 c^2 g_{00}(g_{00}\,\mathrm{d}t^2 - \mathrm{d}l^2)^{-1/2}\,\mathrm{d}t \tag{1}$$

where c is the velocity of light in empty gravitation-free space, g_{00} the (00)-component of the metric tensor and $\mathrm{d}l$ an invariant infinitesimal element of spatial displacement.

If we introduce the velocity

$$v = \mathrm{d}l/\sqrt{g_{00}}\,\mathrm{d}t \tag{2}$$

of the particle, measured in terms of the proper time, that is, by an observer located at a given point, then we obtain for the energy,

$$E = \frac{m_0 c^2 \sqrt{g_{00}}}{(1 - v^2/c^2)^{1/2}}. \tag{3}$$

Landau and Lifshitz have also shown that the expression (3) remains valid for a

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stationary gravitational field if v is given by

$$\frac{c\,\mathrm{d}l}{\sqrt{g_{00}}(\mathrm{d}t+g_{0\sigma}\,\mathrm{d}x^{\sigma}/g_{00})}, \qquad \sigma=1,2,3. \tag{4}$$

In terms of $\sqrt{g_{00}(O)}\,\mathrm{d}t$, the proper time of an observer O located at $x^{\sigma}(O)$, the expressions (3) and (4) give for the velocity

$$v(O)=\frac{\mathrm{d}l}{\sqrt{g_{00}(O)}\,\mathrm{d}t}=\left[\frac{g_{00}}{g_{00}(O)}\left(1-\frac{g_{00}}{\kappa^2}\right)\right]^{1/2}\left(1+\frac{g_{0\sigma}}{g_{00}}\frac{\mathrm{d}x^{\sigma}}{\mathrm{d}t}\right) \tag{5}$$

where $\kappa=E/m_0c^2$.

Lewis (1932) has given the following line-element for an infinite cylinder rotating about its axis

$$\mathrm{d}s^2=f\,\mathrm{d}t^2-\rho(\mathrm{d}r^2+\mathrm{d}z^2)-l\,\mathrm{d}\phi^2+2m\,\mathrm{d}\phi\,\mathrm{d}t \tag{6}$$

where f, ρ, l and m are given by the expressions

$$\begin{aligned} f&=(1-\omega^2)^{-1}(r^{2k}-\omega^2r^{2-2k}),\\ l&=(1-\omega^2)^{-1}(r^{2-2k}-\omega^2r^{2k}),\\ m&=(1-\omega^2)^{-1}\omega(r^{2-2k}-r^{2k}),\\ \rho&=D^2r^{2k(k-1)}. \end{aligned} \tag{7}$$

Here D and k are constants related to mass per unit length of the cylinder and ω represents its angular velocity.

The expression for the velocity of material particles in the field is given from (5) by

$$\left(\frac{\mathrm{d}l}{\sqrt{f(O)}\,\mathrm{d}t}\right)=\left[\frac{f}{f(O)}\left(1-\frac{f}{\kappa^2}\right)\right]^{1/2}\left(1+\frac{m}{f}\frac{\mathrm{d}\phi}{\mathrm{d}t}\right). \tag{8}$$

Equation (8) shows that the velocity of a material particle in the field of a rotating cylinder receives a contribution from the angular velocity $\mathrm{d}\phi/\mathrm{d}t$. This statement may easily be seen to hold for photons also.

Next we consider the geodesic equations for particles in this field. They are

$$f\dot{t}^2-\rho(\dot{r}^2+\dot{z}^2)-l\dot{\phi}^2+2m\dot{\phi}\dot{t}=\delta,$$

$$\ddot{z}+\frac{\rho'}{\rho}\dot{r}\dot{z}=0,$$

$$\ddot{\phi}+\left[\frac{fl'}{r^2}-\frac{mm'}{r^2}\right]\dot{r}\dot{\phi}-\left[\frac{fm'}{r^2}+\frac{mf'}{r^2}\right]\dot{r}\dot{t}=0,$$

$$\ddot{t}+\left[\frac{ml'}{r^2}+\frac{lm'}{r^2}\right]\dot{r}\dot{\phi}-\left[\frac{mm'}{r^2}-\frac{lf'}{r^2}\right]\dot{r}\dot{t}=0,$$

where dots represent differentiation with respect to s for material particles and with respect to an affine parameter for photons while primes refer to differentiation with respect to r. $\delta=+1$ for material particles and $\delta=0$ for photons.

The first integrals of the above equations may be easily obtained by a technique due to Adler *et al* (1975) as follows:

$$f\dot{t}^2-\rho(\dot{r}^2+\dot{z}^2)-l\dot{\phi}^2+2m\dot{\phi}\dot{t}=\delta, \tag{9}$$

$$f\dot{t}+m\dot{\phi}=A, \tag{10}$$

$$\rho\dot{z}=B, \tag{11}$$

$$-m\dot{t}+l\dot{\phi}=C \tag{12}$$

where A, B and C are constants of integration. A and C represent respectively the total energy and the total angular momentum of the test particle.

From (10) and (12),

$$\dot{\phi}=\frac{Am+Cf}{fl+m^2}, \tag{13}$$

$$\dot{t}=\frac{Al-Cm}{fl+m^2}. \tag{14}$$

For the trajectory of the particles in a constant z plane, we will have to integrate the equation,

$$\left(\frac{dr}{d\phi}\right)^2=\frac{1}{\rho}\left[f\frac{(Al-Cm)^2}{(Am+Cf)^2}-l+2m\frac{(Al-Cm)}{(Am+Cf)}-\frac{\delta r^4}{(Am+Cf)^2}\right] \tag{15}$$

Table 1. (r_c is the upper or lower limit (cut-off value) of r_0)

	k	Material particles					Photons		
		for $A^2>1$			for $A^2<1$				
		α	β	y_1	α	β	α	β	y_1
	$\frac{1}{4}$	$A\omega$	$\frac{A}{\omega}$	$\frac{1}{\omega^2}$	$A\omega$	$\frac{A\omega(1-r_c)}{(1-\omega^2 r_c)}$	$A\omega$	$\frac{A}{\omega}$	$\frac{1}{\omega^2}$
	$\frac{1}{2}$	A	$-A$	—	A	A^2	A	$-A$	—
$\omega<1$	$\frac{3}{4}$	$\frac{A}{\omega}$	$\frac{A\omega(r_c-1)}{(r_c-\omega^2)}$	ω^2	$\frac{A}{\omega}$	$\frac{A}{\omega}\cdot\frac{(r_c-1)}{(r_c/\omega^2-1)}$	$\frac{A}{\omega}$	$A\omega$	ω^2
	1	$\frac{A}{\omega}$	$\frac{A\omega(r_c^2-1)}{(r_c^2-\omega^2)}$	ω	$\frac{A}{\omega}$	$\frac{A}{\omega}\cdot\frac{(r_c^2-1)}{(r_c^2/\omega^2-1)}$	$\frac{A}{\omega}$	$A\omega$	ω
	2	$\frac{A}{\omega}$	$\frac{A\omega(r_c^4-r_c^{-2})}{(r_c^4-\omega^2 r_c^{-2})}$	$\omega^{1/3}$	$\frac{A}{\omega}$	$\frac{A}{\omega}\cdot\frac{(r_c^6-1)}{(r_c^6/\omega^2-1)}$	$\frac{A}{\omega}$	$A\omega$	$\omega^{1/3}$
	$\frac{1}{4}$	$A\omega$	$\frac{A\omega(1-r_c)}{(1-\omega^2 r_c)}$	$\frac{1}{\omega^2}$	$A\omega$	$\frac{A\omega(1-r_c)}{(1-\omega^2 r_c)}$	$A\omega$	$\frac{A}{\omega}$	$\frac{1}{\omega^2}$
	$\frac{1}{2}$	A	$-A$	—	A	A^2	A	$-A$	—
$\omega>1$	$\frac{3}{4}$	$\frac{A}{\omega}$	$A\omega$	ω^2	$\frac{A}{\omega}$	$\frac{A}{\omega}\cdot\frac{(r_c-1)}{(r_c/\omega^2-1)}$	$\frac{A}{\omega}$	$A\omega$	ω^2
	1	$\frac{A}{\omega}$	$A\omega$	ω	$\frac{A}{\omega}$	$\frac{A}{\omega}\cdot\frac{(r_c^2-1)}{(r_c^2/\omega^2-1)}$	$\frac{A}{\omega}$	$A\omega$	ω
	2	$\frac{A}{\omega}\cdot\frac{(1-r_c^6)}{(1-r_c^6/\omega^2)}$	$A\omega$	$\omega^{1/3}$	$\frac{A}{\omega}$	$\frac{A}{\omega}\cdot\frac{(r_c^6-1)}{(r_c^6/\omega^2-1)}$	$\frac{A}{\omega}$	$A\omega$	$\omega^{1/3}$

obtained by using equations (9), (13) and (14). For circular orbits, we have from this equation

$$f(Al - Cm)^2 - l(Am + Cf)^2 + 2m(Al - Cm)(Am + Cf) - \delta r^4 = 0. \qquad (16)$$

For a general trajectory, it seems difficult to integrate equation (15). We can, however, obtain restrictions imposed upon the motion of particles in a Lewis field in an alternative way as follows.

At the turning point r_0 ($\dot{r} = 0$) in a constant z plane, (9) reduces to

$$f_0\dot{t}^2 - l_0\dot{\phi}^2 + 2m_0\dot{\phi}\dot{t} = \delta.$$

Putting (13) and (14) in the above equation, we get,

$$C^2 f_0 + 2ACm_0 + (\delta r_0^2 - A^2 l_0) = 0. \qquad (17)$$

Substituting for m_0 and f_0 from (7) and (17), we get

$$C_\pm = \frac{-Ay^2\omega(r_0^{2-2k} - r_0^{2k}) \pm r_0[A^2 - \delta y^2(r_0^{2k} - \omega^2 r_0^{2-2k})]^{1/2}}{y^2(r_0^{2k} - \omega^2 r_0^{2-2k})}, \qquad (18)$$

where $y^2 = (1 - \omega^2)^{-1}$. C_+ and C_- represent angular momentum of the text particle at the turning point r_0.

We now consider variations of C_+ and C_- for (i) material particles ($\delta = +1$) with $A^2 > 1$ and $A^2 < 1$, (ii) photons ($\delta = 0$). Such variations are considered for various values of k and ω and for this purpose we make use of table 1. It should be noted that since we are interested only in the restriction caused to the motion of particles in a cylindrically symmetric stationary gravitational field, we draw the curves *only broadly* (not exactly). Continuous curves are for material particles while the dotted ones are for photons.

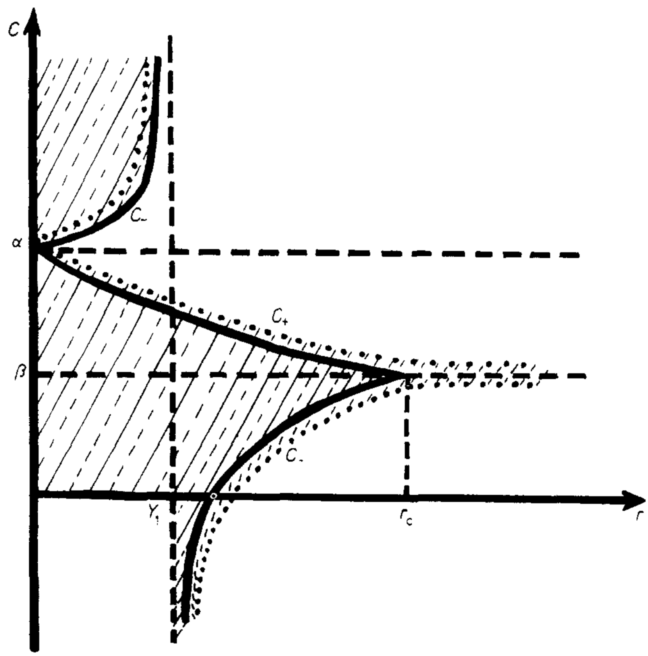

Figure 1. $k > \frac{1}{2}$ with $\omega < 1$ and $k < \frac{1}{2}$ with $\omega > 1$ for material particles ($A^2 > 1$) and photons.

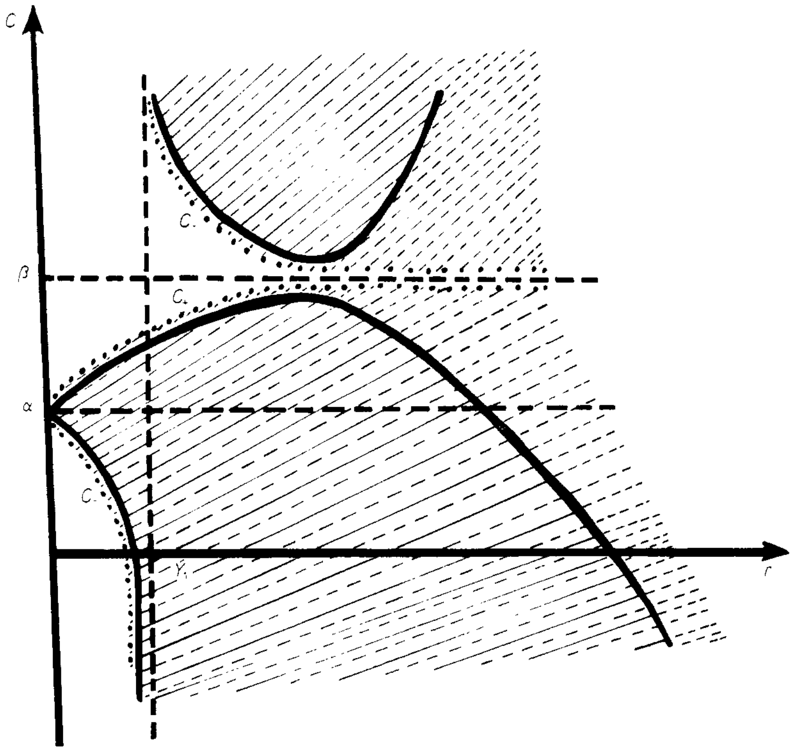

Figure 2. $k<\frac{1}{2}$ with $\omega<1$ and $\frac{1}{2}<k<1$ with $\omega>1$ for material particles ($A^2>1$) and photons.

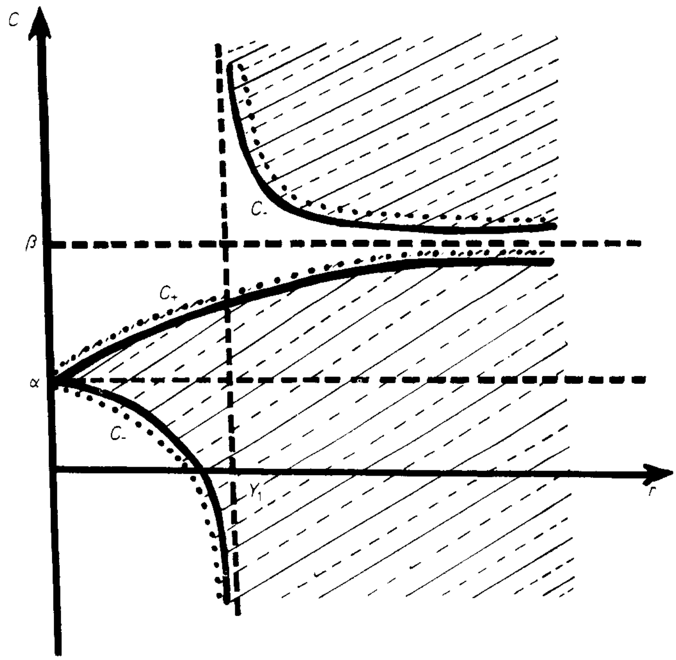

Figure 3. $k=1$ with $\omega>1$ for material particles ($A^2>1$) and photons.

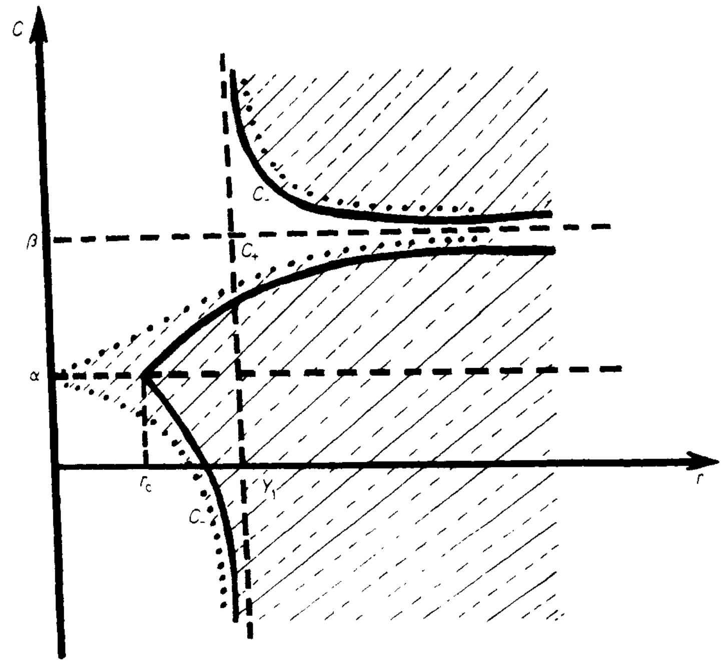

Figure 4. $k > 1$ with $\omega > 1$ for material particles ($A^2 > 1$) and photons.

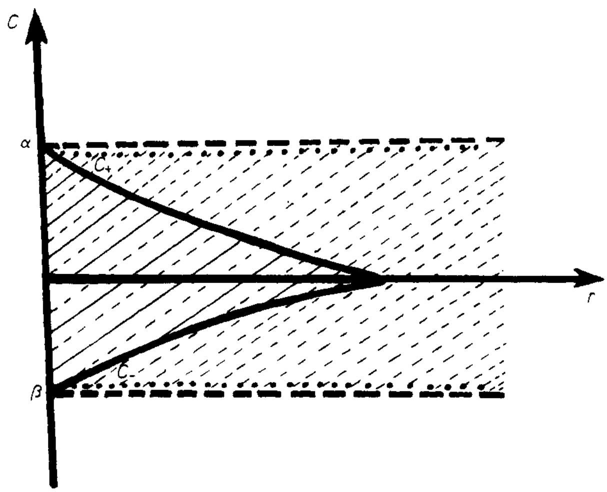

Figure 5. $k = 1$ with $\omega < 1$ and $k = \frac{1}{2}$ with $\omega > 1$ for material particles ($A^2 > 1$) and photons.

3. Discussion

First, we would like to point out that in our analysis particles are restricted to regions where

$$C^2 f + 2ACm + \delta r^2 - A^2 l + \rho r^2 \dot{r}^2 = 0 \tag{19}$$

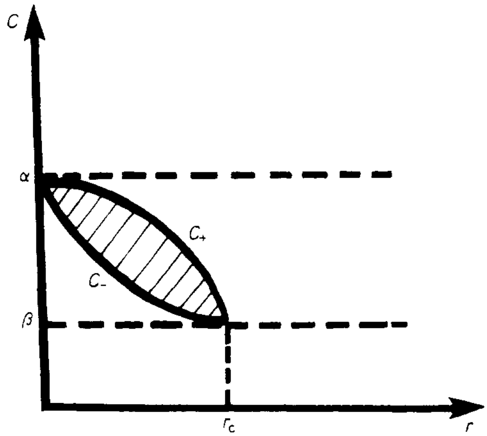

Figure 6. $k > \frac{1}{2}$ with $\omega > 1$ and $k < \frac{1}{2}$ with $\omega < 1$ for material particles ($A^2 < 1$) only.

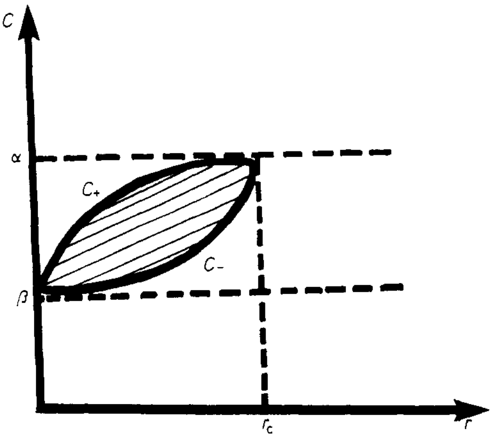

Figure 7. $k = \frac{1}{2}$ with $\omega < 1$ and $k = \frac{1}{2}$ with $\omega > 1$ for material particles ($A^2 < 1$) only.

holds. This equation shows that for a given value of r, particles will have C as follows:

$$C_+ > C > C_- \quad \text{for } f > 0, \tag{20}$$

either $\quad C > C_- > C_+$
or $\quad C_- > C_+ > C$

$$\left.\begin{matrix} C > C_- > C_+ \\ C_- > C_+ > C \end{matrix}\right\} \text{for } f < 0. \tag{21}$$

These regions have been shaded with continuous lines for material particles and with dashes for photons in all the figures.

If v_s is the proper frequency of light emitted by a source at rest at x_s^μ and v_0 is the frequency of the light observed at x_0^μ, then these two frequencies are related by the equation (Adler *et al*)

$$v_0 = v_s[g_{00}(x_s^\mu)/g_{00}(x_0^\mu)]^{1/2}. \tag{22}$$

Obviously the surface $r = y_1$ (where $g_{00}(x_s^\mu)$ i.e., f is zero) is one of infinite redshift. It is easy to check that this is not a null hypersurface; the normal vector and its norm are respectively

$$n_\alpha = (0, 1, 0, 0), \tag{23}$$

$$n^\alpha n_\alpha = -1/\rho. \tag{24}$$

Since the norm is clearly negative, the infinite red-shift surface will pass material particles in both directions and is not therefore a one-way membrane like the Schwarzschild surface (see Adler *et al*).

Finally, we draw some conclusions from the figures.

(*i*) *Material particles.* Figures 1, 2, 5, 6 and 7 show that material particles cannot escape from the field of the rotating cylinder. They are trapped. The possibility of escape appears in figures 3 (for $A^2 > 1$, $k = 1$ with $\omega > 1$) and 4 (for $A^2 > 1$, $k > 1$ with $\omega > 1$) only. In these cases, particles may approach the cylinder from infinity also.

(*ii*) *Photons.* The possibility for escape of photons from the field of a rotating cylinder is shown in all the relevant figures (i.e., Figures 1–5). However, this possibility is restricted only to values of C close to β in figure 1 and between α and β in figure 5. In all these cases, photons may approach the cylinder from infinity also.

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References

Adler R, Bazin M and Schiffer M 1975 *Introduction to General Relativity* (New York: McGraw-Hill)
Banerjee A 1968 *J. Phys. A: Math. Gen.* **1** 495
De Felice F and Calvani M 1972 *Nuovo Cim.* B **10** 447
Krori K D and Chaudhury Sumita 1978 *Ind. J. Pure Appl. Math.* to be published
Landau L D and Lifshitz E M 1971 *The classical theory of fields* (Massachusetts: Addison-Wesley)
Lewis T 1932 *Proc. R. Soc.* A **136** 176
Synge J L 1966 *Mon. Nat. R. Astr. Soc.* **131** 463